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LINEAR HIGH RESOLUTION FREQUENCY-WAVENUMBER ANALYSIS

by

Eugene Smart

Abstract

The marginal success of the several high-resolution frequency-wavenumber (f-k) techniques to data is cited from the literature. Their ability to resolve signals from two closely spaced sources is not markedly superior to that of ordinary beamforming. Moreover, such nonlinear techniques yield distorted magnitudes and azimuths. The ordinary f-k "spectrum" is shown to be no more than a 1-signal estimator, and the high resolution techniques to be but variations of that 1-signal estimator. In this paper the notion of the wavenumber "spectrum" is set aside. By analogy to the 1-signal estimator (the ordinary f-k spectrum) a linear M-signal estimator is developed. The high resolving power of this technique and the fidelity of its estimates is demonstrated theoretically and by computer examples both real and synthetic.

The frequency-wavenumber spectrum, which is a multi-dimensional equivalent of the ordinary frequency spectrum, is used in the sciences for theoretical and experimental analysis of traveling waves. It was introduced formally into seismology by Burg (5) in an application to data analysis. The ordinary unsmoothed three dimensional frequency-wavenumber spectrum of time series data sampled at discrete points in space is given by

$$P(\omega, \bar{k}) = \left| \frac{1}{N} \sum_{n=1}^{N} \left\{ A_n(\omega) \exp[i\alpha_n(\omega)] \right\} \cdot \exp(i\bar{k} \cdot \bar{r}_n) \right|^2$$

(1)

where

n is the index of the spacial sample points.

$$A_n(\omega) \exp[i\alpha_n(\omega)]$$

is the finite Fourier transform of the nth time series.

$\bar{k}$ is the vector wavenumber.

$\bar{r}_n$ is the vector location of the nth sensor, or sample point.

Each Fourier transform term is equivalent to a sinusoid. For example, the sinusoid for the nth transform at frequency ω has amplitude $A_n(\omega)$ and phase $\alpha_n(\omega)$ (at the center of the time window).

Now $-\bar{k} \cdot \bar{r}_n$ is the phase delay, between the origin

and $\bar{r}_n$, of a plane wave arriving from the azimuth of the vector $\bar{k}$ and traveling at the phase velocity

$$v = \omega / |\bar{k}|$$

So, multiplication of the transform by the kernel $\exp(i\bar{k}\cdot\bar{r}_n)$ has the effect of advancing the sinusoid by just the amount the wave itself had delayed it. Thus the summation in (1), above, is a beam sum, and the f-k spectrum is just the frequency domain equivalent of ordinary beam steering.

When the traveling-wave delays are exactly compensated for by the beam shifting, i.e., when the true $\bar{k}$ of the signal is selected, the sinusoids add up constructively with no interference, and the power, P, is maximized. Within certain limits, then, maxima or peaks in f-k space are treated as indications of the presence of traveling plane waves, and the location and size of the maxima are taken as estimates of the speed, bearing, frequency and power of those signals. If more than one signal is present or if there is noise in the data, though exact determinations are no longer possible, the f-k spectrum is still useful for detecting and estimating signals, again within limits.

One of those limitations is imposed by the finite width of the maxima associated with signals. (9) The case is analogous to that of the ordinary frequency spectrum in which components are represented by peaks of finite width.

Plane wave signal peaks in the $f\text{-}k$ spectrum have a half-power width of the order

$$\Delta k \approx \frac{1}{\Delta x}$$

where Δx is the width, or aperature, of the array of spatial sample points. (4) If two signals in the same time window and frequency band are also close enough in phase velocity and azimuth so their wavenumbers, say $\overline{k}_1$ and $\overline{k}_2$, are such:

$$\left|\overline{k}_1 - \overline{k}_2\right| < \Delta k,$$

then their maxima in the $f\text{-}k$ spectrum are merged and form a single peak. (23) Thus, because the sensor arrays are spatially finite their resolving power is finite. Attempts to increase that power of resolution through data processing technique have required mathematical schemes to reduce the width of the lobe of the signal peak (1-3 6-15, 17, 19). However, the straight-forward geometric appeal of this approach has proved misleading thus far. In such hybrid spectra signal lobe-widths indeed have been narrowed substantially. Nevertheless, when signal pairs approach each other in the k-plane, resolution still fails as the separation nears Δk, to wit, the lobe half-width for the ordinary $f\text{-}k$ spectrum. (2, 11, 13, 15, 20).

Observations of other investigators on the shortcomings of various high-resolution frequency-wavenumber techniques are cited below.

Lintz (1968) finds that the high-resolution f-k spectral technique of Haney (1967) does not significantly improve the capability of a seismic array to detect multiple time-overlap-pint events from different azimuths.

Salat and Sax (1969) experimentally find the high-resolution f-k spectrum of Haney (1967), and that of Capon (1968), (1969) no better at resolving two simultaneously arriving waves than the ordinary f-k spectrum. McCowan and Lintz (1968) call attention to an unrecoverable distortion of the true amplitude spectrum in Haney's technique, and the marked disadvantage of spurious peaks under certain conditions which they regard as the inevitable result of using a high-gain procedure.

Seligson (1970) describes conditions under which Capon's high-resolution technique displays less "angular resolution" than ordinary beamforming. McDonough (1972) concludes that variations in amplitude from sensor to sensor may be expected to produce anomolous behavior in Capon's processor. Of course, just such variation in amplitude from sensor to sensor will result precisely because of the presence of two or more signals.

McDonough offers arguments to show that ordinary beamforming is less susceptible to instability resulting from small signal modeling errors than all other array processors.

Haney, too, notes that in the processor he describes (1967) variation in amplitude from sensor-to-sensor could distort the spectrum beyond recognition. He remedies this difficulty by forcing the same amplitude upon each input channel, thus destroying the very amplitude information that would be indicative of the presence of two or more signals.

Woods (1973) concludes that given favorable conditions, the resolving power of the maximum-likelihood f-k spectrum can be effectively infinite, but, disappointingly, offers computer examples on synthetic data in which the input signal pairs are well spaced to begin with (they are separated by a distance of 0.9 of the main-lobe half-width). Cox (1973) also offers theory suggesting that given arbitrarily high signal-to-noise ratios arbitrarily fine resolution should be possible, but he does not offer a method.

It may be argued that the limited resolving power of the several high-resolution techniques results from the wavenumber spectrum being in reality a 1-signal estimator. Indeed, the ordinary f-k "spectrum" is a least squares estimator for fitting given data to a single plane wave, as shown further

on. In routine automated processing of the LASA LP data Mack and Smart (1973) found the ordinary spectrum useful for estimating only one signal at a time. Estimates of a possible second signal were made by recomputing the wavenumber spectrum after the first (and larger) estimate had been subtracted from the data. They call this process stripping; it is useful, of course, only for estimating signals separated by about the reciprocal of the array diameter or more. At that, such estimates of a pair of signals are not optimum, but first order approximations.

Properly, the f-k spectrum is defined only for signals of infinite spacial extent traversing infinitely large arrays. The effect of a signal of wavenumber $\overline{k}_o$ is then confined to the point $\overrightarrow{k}_o$ in the spectrum. Approximations to this definition are useful if the dimensions of signals and arrays are sufficiently large. Failing that, the "spectrum" reduces to a 1-signal estimator as noted. While the high-resolution techniques do attempt to extend the effective array diameter, they all test the wavenumber space with a 1-signal probe, as in the ordinary f-k spectrum.

It is proposed here to set aside the notion of a spectrum. Rather we will extend the 1-signal estimator to an M-signal estimator thus to permit the simultaneous removal

of the effects of one signal from the estimate of another and so achieve true high-resolution. t the same time, use of beamforming (in the k -plane) to estimate each of the M signals will preserve the stability and estimate fidelity of the ordinary f-k spectrum.

In the following discussion a 1-signal least squares estimator is developed and is identified with the ordinary f-k spectrum. Analogy to the 1-signal estimator is used to develop an M-signal estimator.

Conventional Frequency-Wavenumber Analysis

In the conventional frequency-wavenumber spectrum (ordinary or high-resolution) a single plane wave is hypothesized at each frequency. That model is then tested over the wavenumber space of interest. One attempts to minimize the error

$$\epsilon = \sum_{n=1}^{N} \left| U_n - A e^{i \vec{k} \cdot \vec{r}_n} \right|^2$$

by varying A and $\vec{k}$ where

U_n	are the complex Fourier series terms (for the given frequency)
n	is the sensor, or channel, index
N	is the total number of sensors
$\vec{r}_n$	are the location vectors of the sensors
A	is the complex Fourier series term for the hypothesized plane wave (at the given frequency)
$\vec{k}$	is the wavenumber of the hypothetical plane wave (at that given frequency)

$$A e^{i\vec{k}\cdot\vec{r}_n}, \quad n = 1, \ldots, N \text{ is the}$$

model, i.e., the hypothesized plane wave.

Note that also one can write ϵ as

$$\epsilon = \sum_{n=1}^{N} \left| U_n e^{-i\vec{k}\cdot\vec{r}_n} - A \right|^2$$

since $\left| e^{-i\vec{k}\cdot\vec{r}_n} \right| = 1$.

For a given $\vec{k}$, ϵ is minimized by setting A to

$$A = \frac{1}{N} \sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n}$$

which is shown by the following:

Let $\qquad a_n + i c_n \equiv U_n e^{-i\vec{k}\cdot\vec{r}_n}$

and $\qquad a + i c \equiv A$

Then

$$\epsilon = \sum_{n=1}^{N} \left| (a_n - a) + i(c_n - c) \right|^2$$

$$= \sum_{n=1}^{N} (a_n - a)^2 + (c_n - c)^2$$

Take partial derivatives:

$$\frac{\partial \epsilon}{\partial a} = -2 \sum_{n=1}^{N} (a_n - a); \quad \frac{\partial \epsilon}{\partial c} = -2 \sum_{n=1}^{N} (c_n - c)$$

$$\frac{\partial^2 \mathcal{E}}{\partial a^2} = \frac{\partial^2 \mathcal{E}}{\partial c^2} = 2N$$

Setting $\quad \dfrac{\partial \mathcal{E}}{\partial a} = \dfrac{\partial \mathcal{E}}{\partial c} = 0 \; ,$

$$a = \frac{1}{N} \sum_{n=1}^{N} a_n \qquad\qquad c = \frac{1}{N} \sum_{n=1}^{N} c_n$$

and

$$a + ic = A = \frac{1}{N} \sum_{n=1}^{N} (a_n + ic_n) = \frac{1}{N} \sum_{n=1}^{N} U_n e^{-i \vec{k}\cdot \vec{r}_n}$$

So, minimized with respect to A,

$$\mathcal{E} = \sum_{n=1}^{N} \left| U_n e^{-i\vec{k}\cdot\vec{r}_n} - \frac{1}{N} \sum_{j=1}^{N} U_j \, e^{-i\vec{k}\cdot\vec{r}_j} \right|^2$$

This expression can be separated into 2 parts, thus:

$$\mathcal{E} = \sum_{n=1}^{N} (a_n - a)^2 + (c_n - c)^2$$

$$= \sum_{n=1}^{N} a_n^2 - 2a_n a + a^2 + c_n^2 - 2c_n c + c^2$$

$$= \sum_{n=1}^{N} (a_n^2 + c_n^2) - 2a\cdot a N + a^2 N - 2c\cdot c N + c^2 N$$

$$= \sum_{n=1}^{N} \left| a_n + ic_n \right|^2 - \frac{1}{N} \left| \sum_{n=1}^{N} a_n + ic_n \right|^2$$

Thus,

$$\mathcal{E} = \sum_{n=1}^{N} \left| U_n \right|^2 - \frac{1}{N} \left| \sum_{n=1}^{N} U_n \, e^{-i\vec{k}\cdot\vec{r}_n} \right|^2$$

The second term is the ordinary frequency-wavenumber spectrum

$$P(f, \vec{k}) = \frac{1}{N} \left| \sum_{n=1}^{N} U_j(f) e^{-i\vec{k}\cdot\vec{r}_n} \right|^2$$

So,

$$\epsilon = \sum_{n=1}^{N} \left| U_n \right|^2 - P(\vec{k})$$

Since ϵ is a squared modulus

$$\epsilon \geqslant 0$$

and

$$\sum_{n=1}^{N} \left| U_n \right|^2 \geqslant 0$$

since it is a sum of squared modulii.

Similarly

$$P(\vec{k}) \geqslant 0$$

Since

$$\sum_{n=1}^{N} \left| U_n \right|^2 - P(\vec{k}) \geqslant 0 ,$$

$$\sum_{n=1}^{N} \left| U_n \right|^2 \geqslant P(\vec{k})$$

So to minimize ϵ one must maximize $P(\vec{k})$.

$$\epsilon(\vec{k}) = \sum_{n=1}^{N} \left| U_n e^{-i\vec{k}\cdot\vec{r}_n} - A \right|^2$$

becomes exactly zero when

$$U_n = e^{i\vec{k}\cdot\vec{r}_n} \cdot \frac{1}{N} \sum_{j=1}^{N} U_j \, e^{-i\vec{k}\cdot\vec{r}_j} = A \, e^{i\vec{k}\cdot\vec{r}_n}$$

$$n = 1, \ldots, N$$

that is, when the data describe a single plane wave exactly.
The smaller $\in(\vec{k})_{min}$ is, in a given situation, the
more likely is the hypothetical plane wave

$$A \, e^{i\vec{k}\cdot\vec{r}}$$

because the smaller $\in(\vec{k})$ is, the larger the F-statistic
is for the hypothesis. The F-statistic is given by

$$F = (N-1) \cdot P(\vec{k}) \big/ \in(\vec{k})$$

This single plane wave model is often applied in
attempts to analyze a 2-signal case (or a possible 2-signal
case). In such an analysis each signal is treated as if it
existed by itself, the presence of the other being ignored
with consequent distortion of estimates by mutual interfer-
ence. This interference can be serious, and if the two
signals are not separated in k -space by at least the half-
width of the main lobe of the array response, they are likely
to appear as but one signal, their main lobes having coalesced.
Attempts to improve the performance of the single wave hypo-
thesis (in application to the two signal case) have been
made in which the main lobe of the array response has been
slenderized mathematically by alternative methods of esti-

mation of the wavenumber spectrum. The object has been to reduce the main-lobe half-width and so resolve signal pairs which otherwise have coalesced main-lobes indistinguishable from a signal case. These results have been marginal. In the various high-resolution techniques the influence of the one signal on the analysis of the other has been ignored.

Analysis of possible 2-signal cases calls for a 2-signal model, in particular when the 2-signals are known (or suspected) to be so close together as to have their main lobes merged.

As the 1-signal model serves for both the 0- and the 1-signal case, so one might expect a 2-signal model to be effective in all three cases: 0, 1, or 2-signals.

Multiple Signal Frequency-Wavenumber Analysis

By analogy to the 1-signal model, one would expect to solve a 2-signal model by minimizing the error

$$\epsilon = \sum_{n=1}^{N} \left| U_n - A e^{i \vec{k} \cdot \vec{r}_n} - B e^{i \vec{k} \cdot r_n} \right|^2$$

varying $A, \vec{k}, B,$ and $\vec{k}$, where

B is the complex Fourier series term for the second hypothesized plane wave (at the same given frequency)

$\vec{k}$ is the wavenumber of the hypothetical plane wave (at that same given frequency)

There are now two signals to solve for:

$$A e^{i \vec{k} \cdot \vec{r}_n} \quad \text{and} \quad B e^{i \vec{k} \cdot \vec{r}_n}$$

Let

$$T_n = U_n - A e^{i \vec{k} \cdot \vec{r}_n} - B e^{i \vec{k} \cdot \vec{r}_n}$$

then

$$\epsilon = \sum_{n=1}^{N} \left| T_n \right|^2 = \sum_{n=1}^{N} T_n^* T_n$$

Again, let

$$A = a + ic, \quad A^* = a - ic$$

Taking first partial derivatives while noting that

$$\frac{\partial A}{\partial a} = \frac{\partial A^*}{\partial a} = 1 \quad \text{and} \quad \frac{\partial A}{\partial c} = -\frac{\partial A^*}{\partial c} = i \quad ,$$

$$\frac{\partial \epsilon}{\partial a} = \sum_{n=1}^{N} T_n^*(-e^{i\vec{k}\cdot\vec{r}_n}) + T_n(-e^{-i\vec{k}\cdot\vec{r}_n})$$

and

$$\frac{\partial \epsilon}{\partial c} = i\sum_{n=1}^{N} T_n^*(-e^{i\vec{k}\cdot\vec{r}_n}) + T_n(e^{-i\vec{k}\cdot\vec{r}_n})$$

Setting

$$\frac{\partial \epsilon}{\partial a} = \frac{\partial \epsilon}{\partial c} = 0$$

as in the 1-signal case,

$$\frac{\partial \epsilon}{\partial a} + i\frac{\partial \epsilon}{\partial c} = -2\sum_{n=1}^{N} T_n e^{-i\vec{k}\cdot\vec{r}_n} = 0$$

Therefore,

$$A = \frac{1}{N}\sum_{n=1}^{N} (U_n - B e^{i\vec{k}\cdot\vec{r}_n}) \cdot e^{-i\vec{k}\cdot\vec{r}_n}$$

Analogously

$$B = \frac{1}{N}\sum_{n=1}^{N} (U_n - A e^{i\vec{k}\cdot\vec{r}_n}) \cdot e^{-i\vec{k}\cdot\vec{r}_n}$$

In this form A and B are optimized, that is, they produce the minimum value of ϵ for any arbitrary pair of $\overline{k}$ and $\vec{k}$. Adopting the notation:

$$P \equiv \frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\overline{k}\cdot\vec{r}_n}$$

$$Q \equiv \frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n}$$

$$E \equiv \frac{1}{N}\sum_{n=1}^{N} e^{i(\overline{k}-\vec{k})\cdot\vec{r}_n}$$

one may write simply:

$$A = P - B \cdot E \quad \text{and} \quad B = Q - A \cdot E^*$$

Rearranging to solve A and B simultaneously

$$P = A + B \cdot E$$

$$Q = A \cdot E^* + B$$

$$A = \frac{\begin{vmatrix} P & E \\ Q & 1 \end{vmatrix}}{\begin{vmatrix} 1 & E \\ E^* & 1 \end{vmatrix}}, \qquad B = \frac{\begin{vmatrix} 1 & P \\ E^* & Q \end{vmatrix}}{\begin{vmatrix} 1 & E \\ E^* & 1 \end{vmatrix}}$$

$$A = (P - QE)/(1 - E^*E)$$

$$B = (Q - PE^*)/(1 - E^*E)$$

Written out at length,

$$A = \frac{\frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n} - \frac{1}{N^2}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n} \sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n}}{1 - \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n} \frac{1}{N}\sum_{j=1}^{N} e^{-i(\vec{k}-\vec{k})\cdot\vec{r}_j}}$$

and B is similar in form.

Introducing a factor of $1/N$ into $\in$

$$\in = \frac{1}{N}\sum_{n=1}^{N} T_n^* T_n$$

$$= \frac{1}{N}\sum_{n=1}^{N} \left(U_n^* - A^* e^{-i\vec{k}\cdot\vec{r}_n} - B^* e^{-i\vec{k}\cdot\vec{r}_n} \right)$$
$$\times \left(U_n - A e^{i\vec{k}\cdot\vec{r}_n} - B e^{i\vec{k}\cdot\vec{r}_n} \right)$$

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n$$

$$-\left(A^* \frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r_n}} + A\frac{1}{N}\sum_{n=1}^{N} U_n^* e^{i\vec{k}\cdot\vec{r_n}}\right)$$

$$-\left(B^* \frac{1}{N}\sum_{n=1}^{N} U_n e^{i\vec{k}\cdot\vec{r_n}} + B\frac{1}{N}\sum_{n=1}^{N} U_n^* e^{i\vec{k}\cdot\vec{r_n}}\right)$$

$$+\left(A^* A + B^* B\right)$$

$$+\left(A^* B \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r_n}} + AB^* \frac{1}{N}\sum_{n=1}^{N} e^{-i(\vec{k}-\vec{k})\cdot\vec{r_n}}\right)$$

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - \left(A^* P + AP^*\right) - \left(B^* Q + BQ^*\right)$$

$$+\left(A^* A + B^* B\right) + \left(A^* BE + AB^* E^*\right)$$

Rearranging the terms in ϵ,

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - \left(A^* P + AP^*\right) - \left(B^* Q + BQ^*\right)$$
$$+ A^*(A + BE) + B^*(AE^* + B)$$

and recalling that

$$P = A + BE \qquad \text{and} \qquad Q = AE^* + B$$

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - \left(AP^* + BQ^*\right)$$

Further　substituting

$$A = P - BE \quad \text{and} \quad B = Q - AE^{*}$$

$$\varepsilon = \frac{1}{N}\sum_{n=1}^{N}|U_n|^2 - \frac{\left(P^{*}P + Q^{*}Q - PQ^{*}E^{*} - P^{*}QE\right)}{1 - E^{*}E}$$

or, written out,

$$\varepsilon = \frac{1}{N}\sum_{n=1}^{N}|U_n|^2$$

$$- \frac{\frac{1}{N}\sum_{n=1}^{N}\left|e^{i\vec{k}\cdot\vec{r}_n}\frac{1}{N}\sum_{j=1}^{N}U_j e^{-i\vec{k}\cdot\vec{r}_j} - e^{i\vec{k}\cdot\vec{r}_n}\frac{1}{N}\sum_{j=1}^{N}U_j e^{-i\vec{k}\cdot\vec{r}_j}\right|^2}{1 - \left|\frac{1}{N}\sum_{n=1}^{N}e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n}\right|^2}$$

The identity of these last 2 equations may be demonstrated by noting that the numerator (above) equals

$$\frac{1}{N}\sum_{n=1}^{N}\left|e^{i\vec{k}\cdot\vec{r}_n}P - e^{i\vec{k}\cdot\vec{r}_n}Q\right|^2$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left(e^{-i\vec{k}\cdot\vec{r}_n}P^{*} - e^{-i\vec{k}\cdot\vec{r}_n}Q^{*}\right)\left(e^{i\vec{k}\cdot\vec{r}_n}P - e^{i\vec{k}\cdot\vec{r}_n}Q\right)$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left\{P^{*}P + Q^{*}Q - PQ^{*}e^{-i(\vec{k}-\vec{k})\cdot\vec{r}_n} - P^{*}Q e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n}\right\}$$

$$= P^{*}P + Q^{*}Q - PQ^{*}E^{*} - P^{*}QE$$

Since ε is a sum of squares, by definition it must
be non-negative everywhere. Therefore the second of the
2 terms in ε, above, must always be

$$\leq \frac{1}{N} \sum_{n=1}^{N} |U_n|^2$$

Thus to minimize $\mathcal{E}$, one must maximize

$$\frac{\frac{1}{N} \sum_{n=1}^{N} \left| e^{i\vec{k}\cdot\vec{r}_n} \cdot \frac{1}{N} \sum_{j=1}^{N} U_j e^{-i\vec{k}\cdot\vec{r}_j} - e^{i\vec{k}\cdot\vec{r}_n} \cdot \frac{1}{N} \sum_{j=1}^{N} U_j e^{-i\vec{k}\cdot\vec{r}_j} \right|^2}{1 - \left| \frac{1}{N} \sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n} \right|^2}$$

This is the 2-signal test, analogous to the ordinary frequency-wavenumber spectrum, which is the 1-signal test. However, it is more convenient to retain the form

$$AP^* + BQ^*$$

This 2-signal f-k "spectrum" then is computed from 3 beams (as the ordinary f-k spectrum is computed from 1 beam). The beams are

$$P = \frac{1}{N} \sum_{r=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n}$$
the mean of the data transforms that have been beamed to $\vec{k}$ (one of the two wavenumber variables),

$$Q = \frac{1}{N} \sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n}$$
the mean of the data transforms after beaming to $\vec{k}$ (the other wavenumber variable),

$$E = \frac{1}{N} \sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n}$$
which is the (complex) array response

This 2-signal test is solved as is the ordinary f-k spectrum, numerically, by searching the wavenumber space of

interest. Now, however, there are 4 dimensions to search, over which to test the error criterion.

It is instructive to submit a known pair of pure, noiseless signals to the 2-signal test to illustrate the function of the elements of the expression:

Let
$$U_n = F e^{i \vec{k} \cdot \vec{r_n}} + G e^{i \vec{\tilde{k}} \cdot \vec{r_n}}, \quad n = 1, \ldots, N$$

Beaming them exactly to $\vec{k}$ and $\vec{\tilde{k}}$ (since these are known in this special case),

$$P = \frac{1}{N} \sum_{n=1}^{N} \left(F e^{i \vec{k} \cdot \vec{r_n}} + G e^{i \vec{\tilde{k}} \cdot \vec{r_n}} \right) \cdot e^{-i \vec{k} \cdot \vec{r_n}}$$

$$= \frac{1}{N} \sum_{n=1}^{N} F + G e^{i (\vec{\tilde{k}} - \vec{k}) \cdot \vec{r_n}}$$

$$= F + G \cdot \frac{1}{N} \sum_{n=1}^{N} e^{i (\vec{\tilde{k}} - \vec{k}) \cdot \vec{r_n}} = F + GE$$

and

$$Q = \frac{1}{N} \sum_{n=1}^{N} \left(F e^{i \vec{k} \cdot \vec{r_n}} + G e^{i \vec{\tilde{k}} \cdot \vec{r_n}} \right) \cdot e^{-i \vec{\tilde{k}} \cdot \vec{r_n}}$$

$$= F E^* + G$$

Then

$$A = (P + QE)/(1 - E^*E)$$
$$= (F + GE - (FE^* + G)E)/(1 - E^*E)$$
$$= (F + GE - FE^*E - GE)/(1 - E^*E)$$
$$= F(1 - E^*E)/(1 - E^*E) = F$$

$$B = \left(FE^* + G - (F + GE)E^* \right)/(1 - E^*E)$$

$$B = G(1 - E^*E)/(1 - E^*E) = G$$

Thus

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N}\left| U_n - Ae^{i\vec{k}\cdot\vec{r}_n} - Be^{i\vec{k}\cdot\vec{r}_n}\right|^2$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left| Fe^{i\vec{k}\cdot\vec{r}_n} + Ge^{i\vec{k}\cdot\vec{r}_n} - Fe^{i\vec{k}\cdot\vec{r}_n} - Ge^{i\vec{k}\cdot\vec{r}_n}\right|^2$$

$$\epsilon = 0$$

This little exercise clarifies a bit the function of the array

response, E, in the signal models A and B.

The development of the 2-signal test, of course,

suggests the derivation of a 3-signal test, by analogy:

First, the form of the test would be, analogously,

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N}\left| U_n - Ae^{i\vec{k}\cdot\vec{r}_n} - Be^{i\vec{k}\cdot\vec{r}_n} - Ce^{i\vec{k}\cdot\vec{r}_n}\right|^2$$

Introducing the notation

$$R \equiv \frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n} \qquad E_1 \equiv \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n}$$

$$E_2 \equiv \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n} \qquad E_3 \equiv \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n}$$

and expanding ϵ:

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N}\left(U_n^* - A^* e^{-i\vec{k}\cdot\vec{r}_n} - B^* e^{-i\vec{k}\cdot\vec{r}_n} - C^* e^{-i\vec{k}\cdot\vec{r}_n}\right)$$

$$\times \left(U_n - Ae^{i\vec{k}\cdot\vec{r}_n} - Be^{i\vec{k}\cdot\vec{r}_n} - Ce^{i\vec{k}\cdot\vec{r}_n}\right)$$

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n$$

$$-\left(A^* \frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n} + A \frac{1}{N}\sum_{n=1}^{N} U_n^* e^{i\vec{k}\cdot\vec{r}_n}\right)$$

$$-\left(B^* \frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n} + B \frac{1}{N}\sum_{n=1}^{N} U_n^* e^{i\vec{k}\cdot\vec{r}_n}\right)$$

$$-\left(C^* \frac{1}{N}\sum_{n=1}^{N} U_n e^{-i\vec{k}\cdot\vec{r}_n} + C \frac{1}{N}\sum_{n=1}^{N} U_n^* e^{i\vec{k}\cdot\vec{r}_n}\right)$$

$$+\left(A^*A + B^*B + C^*C\right)$$

$$+\left(A^*B \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n} + AB^* \frac{1}{N}\sum_{n=1}^{N} e^{-i(\vec{k}-\vec{k})\cdot\vec{r}_n}\right)$$

$$+\left(B^*C \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n} + BC^* \frac{1}{N}\sum_{n=1}^{N} e^{-i(\vec{k}-\vec{k})\cdot\vec{r}_n}\right)$$

$$+\left(A^*C \frac{1}{N}\sum_{n=1}^{N} e^{i(\vec{k}-\vec{k})\cdot\vec{r}_n} + AC^* \frac{1}{N}\sum_{n=1}^{N} e^{-i(\vec{k}-\vec{k})\cdot\vec{r}_n}\right)$$

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n$$

$$-\left(A^*P + AP^*\right) - \left(B^*Q + BQ^*\right) - \left(C^*R + CR^*\right)$$

$$+\left(A^*A + B^*B + C^*C\right)$$

$$+\left(A^*BE_1 + AB^*E_1^*\right) + \left(B^*CE_3 + BC^*E_3^*\right) + \left(A^*CE_2 + AC^*E_2^*\right)$$

Now noting that in the 2-signal test

$$P = A + BE \quad \text{and}$$
$$Q = AE^* + B$$

so that

$$A = \frac{\begin{vmatrix} P & E \\ Q & 1 \end{vmatrix}}{\begin{vmatrix} 1 & E \\ E^* & 1 \end{vmatrix}} \qquad\qquad B = \frac{\begin{vmatrix} 1 & P \\ E^* & Q \end{vmatrix}}{\begin{vmatrix} 1 & E \\ E^* & 1 \end{vmatrix}}$$

one recognizes that, in the 3-signal test,

$$P = A + BE_1 + CE_2$$
$$Q = AE_1^* + B + CE_3$$
$$R = AE_2^* + BE_3^* + C$$

and, defining

$$\text{den} \equiv \begin{vmatrix} 1 & E_1 & E_2 \\ E_1^* & 1 & E_3 \\ E_2^* & E_3^* & 1 \end{vmatrix}$$

$$A = \begin{vmatrix} P & E_1 & E_2 \\ Q & 1 & E_3 \\ R & E_3^* & 1 \end{vmatrix} \cdot \text{den}^{-1} \qquad , \text{ etc., or}$$

$$A = [\, P(1 - E_3^* E_3) + Q(E_3^* E_2 - E_1) + R(E_1 E_3 - E_2)]/\text{den}$$
$$B = [\, P(E_2^* E_3 - E_1^*) + Q(1 - E_2^* E_2) + R(E_1^* E_2 - E_3)]/\text{den}$$
$$C = [\, P(E_1^* E_3^* - E_2^*) + Q(E_1 E_2^* - E_3^*) + R(1 - E_1^* E_1)]/\text{den}$$
$$\text{den} = 1 - E_1^* E_1 - E_2^* E_2 - E_3^* E_3 + E_1 E_2^* E_3 + E_1^* E_2 E_3^*$$

Now rearranging $\in$,

$$\in = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - (A^* P + B^* Q + C^* R) - (A P^* + B Q^* + C R^*)$$

$$+ A^*(A + B E_1 + C E_2) + B^*(A E_1^* + B + C E_3) + C^*(A E_2^* + B E_3^* + C)$$

and substituting P, Q, and R,

$$\in = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - (A P^* + B Q^* + C R^*)$$

To minimize $\in$, then, one will maximize

$$A P^* + B Q^* + C R^*$$

the 3-signal test, or 3-signal analog to the conventional, 1-signal frequency-wavenumber spectrum. The function is composed of 6 beams: P, Q, and R, the 3 beams of the data, U_n, and E_1, E_2, and E_3, the 3 beams of the array response.

Remembering the 1-signal test (conventional $f-k$ spectrum),

$$\frac{1}{N}\in = \frac{1}{N}\sum_{n=1}^{N} \left| U_n - A e^{i\vec{k}\cdot\vec{r_n}} \right|^2$$

$$= \frac{1}{N}\sum_{n=1}^{N} \left(U_n^* - A^* e^{-i\vec{k}\cdot\vec{r_n}} \right)\left(U_n - A e^{i\vec{k}\cdot\vec{r_n}} \right)$$

we may rewrite it as

$$\frac{1}{N}\in = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - A^* P - A P^* - A^* A$$

$$\left(\text{since} \quad P = \frac{1}{N}\sum_{n=1}^{N} U_n e^{i\vec{k}\cdot\vec{r}_n}\right), \text{ or}$$

$$\frac{1}{N}\in \, = \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - A^*A - AP^* + A^*A$$

$$= \frac{1}{N}\sum_{n=1}^{N} U_n^* U_n - AP^*$$

Thus

$$AP^* = \frac{1}{N}\left(\frac{1}{N}\left|\sum_{n=1}^{N} U_n e^{i\vec{k}\cdot\vec{r}_n}\right|^2\right)$$

is the expression one must maximize in order to minimize

the error. So the $f\text{-}k$ spectrum (for the 1-signal, conven-

tional, case) is

$$AP^* ,$$

and $\qquad AP^* + BQ^*$ is the 2-signal test,

and $\qquad AP^* + BQ^* + CR^*$ is the 3-signal test.

In the 1-signal test

$$A = \frac{P}{1}$$

For the 2-signal test

$$A = \frac{\begin{vmatrix} P & E_1 \\ Q & 1 \end{vmatrix}}{\begin{vmatrix} 1 & E_1 \\ E_1^* & 1 \end{vmatrix}}, \qquad\qquad B = \frac{\begin{vmatrix} 1 & P \\ E_1^* & Q \end{vmatrix}}{\begin{vmatrix} 1 & E_1 \\ E_1^* & 1 \end{vmatrix}}$$

For the 3-signal test

$$A = \frac{\begin{vmatrix} P & E_1 & E_2 \\ Q & 1 & E_3 \\ R & E_3^* & 1 \end{vmatrix}}{\begin{vmatrix} 1 & E & E \\ E_1^* & 1 & E \\ E_2^* & E_3^* & 1 \end{vmatrix}}, \quad etc.$$

This formalism makes evident the relationship between the successive tests. Thus one may extropolate and directly write the expression for the M-signal test in simple, terse form. For example, the 4-signal test is

$$AP^* + BQ^* + CR^* + DS^*$$

in which S, the sum of the data beamed to yet a 4th point, $\overline{\ast}$, is introduced into the sequence P, Q, and R; and in which

$$A = \frac{\begin{vmatrix} P & E_1 & E_2 & E_4 \\ Q & 1 & E_3 & E_5 \\ R & E_3^* & 1 & E_6 \\ S & E_5^* & E_6^* & 1 \end{vmatrix}}{\begin{vmatrix} 1 & E_1 & E_2 & E_4 \\ E_1 & 1 & E_3 & E_5 \\ E_2^* & E_3^* & 1 & E \\ E_4^* & E_5^* & E_6^* & 1 \end{vmatrix}} , \text{ etc.,}$$

and E_4 is the array response at $(\overline{\ast} - \overline{k})$, E_5, that at $(\overline{\ast} - \overline{k})$, etc.

Note that the four-signal test is computed from 10 beams; 4 beams of the input, U_n, and 6 of the array response. In general, the M-signal test requires M beams of input data (U_n), and $M(M-1)/2$ beams on the array response, for a

total of $M(M+1)/2$ beams to compute the least-squares error at any point in the **2M**-dimensional space. But the beams on the array response are computed from the same complex trigonometric terms that are required for the **M** beams of the input data. So the **M**-signal test requires evaluation of **2MN** sine and cosine terms to compute the error at any point (**N** is the number of sensors in the array). Thus the number of trigonometric terms requiring computation increases linearly with **M**.

It must be noted that a multiple signal test is not everywhere well-behaved, but has a singularity. For example, in the case of the 2-signal test, if

$$\vec{k} \rightarrow \vec{k}$$

so that

$$Q \rightarrow P$$

and

$$E \rightarrow 1$$

$\in$ is undefined. The value it will take on at $\vec{k} = \vec{k}$ depends on the direction from which $\vec{k} \rightarrow \vec{k}$. Though this can, of course, be shown analytically, it is a bit tedious for repetition here. The contoured map of an example (figure   ) displays this characteristic graphically. The contoured function is the 2-signal test

$$A P^* + B Q^*$$

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$$AP^* + BQ^*$$

with $\overrightarrow{k}$ held fixed as $\overrightarrow{k}$ varies over the plane. Note that the contour lines all run together at $\overrightarrow{k} = \overrightarrow{k}$. $\overrightarrow{k}$ may range arbitrarily close to k but must not take on that value exactly. The data in this figure consist of 2 closely spaced signals. The fixed vector, $\overrightarrow{k}$, was set at the peak of their merged main lobes.

One might dismiss this singularity from practical consideration since signals of identical speed and bearing are indistinguishable by array methods. The test for 2 signals at the same wavenumber location is thus unnecessary anyway. But if the 2-signal test, say, is applied to data composed of only 1 signal, must not both the probe vectors approach the same point, i.e., the wavenumber location of the input signal, in order to merge and reduce the function to the 1-signal test? We have seen that when the data, $U_\wedge$, consist of the same number of signals as that for which one is testing, the test performs as expected: the error _is_ minimized at the wavenumber location of those input signals, and the signals _are_ recovered undistorted. Suppose, though, that the 2-signal test, say, is applied to data consisting of just plane wave. Let

$$U_\wedge = F e^{i\overrightarrow{k}\cdot\overrightarrow{r_\wedge}} .$$

$$T_\wedge = U_\wedge - A e^{i\overrightarrow{k}\cdot\overrightarrow{r_\wedge}} - B e^{i\overrightarrow{k}\cdot\overrightarrow{r_\wedge}}$$

in the error expression

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N}|T_n|^2$$

We have to maximize

$$AP^* + BQ^* \ .$$

$$P = F\,\frac{1}{N}\sum_{n=1}^{N}e^{i(\vec{\bar{k}}-\vec{k})\cdot\vec{r}_n}, \text{ and } Q = F\,\frac{1}{N}\sum_{n=1}^{N}e^{i(\vec{\bar{k}}-\vec{\bar{k}})\cdot\vec{r}_n}$$

If $\vec{k}$ goes to $\vec{\bar{k}}$, then

$$P = F, \qquad Q = FE^*$$

and

$$A = (P - QE)\big/(1 - E^*E)$$

becomes

$$A = (F - FE^*E)\big/(1 - E^*E) = F$$

and

$$B = (Q - PE^*)\big/(1 - E^*E) = (FE^* - FE^*)\big/(1 - E^*E)$$
$$= 0$$

and

$$\epsilon = \frac{1}{N}\sum_{n=1}^{N}U_n^* U_n - (AP^* + BQ^*)$$

$$= \frac{1}{N}\sum_{n=1}^{N}(F^* e^{-i\vec{\bar{k}}\cdot\vec{r}_n})(F e^{i\vec{\bar{k}}\cdot\vec{r}_n}) - (F^*F + 0)$$

$$= \frac{1}{N}\sum_{n=1}^{N}F^*F - F^*F = 0$$

When $\vec{k}$ goes to $\vec{\bar{k}}$ the error is minimized, the

signal, F , is recovered undistorted, and the hypothesized

second signal vanishes. This solution is invariant though

$\overline{k}$ be permitted to range over the entire k-plane, excepting the point $\overline{k}$. Thus the 2-signal test does not reduce to the ordinary f-k spectrum in the presence of a single plane wave, and $\overline{k}$ is not required to go to $\overline{k}$ nor would the gradient of $\in$ with respect to $\overline{k}$ lead to $\overline{k}$ (if one were using a steepest descent technique to minimize $\in$).

Numerical Solution of the Multiple Signal Test

One might propose to carry out the numerical solution
of a multiple signal test by a straightforward search of
the entire wavenumber space of interest, as is done in the
computation of the conventional $f-k$ spectrum. But the
multiple signal test may be used in more practical fashion,
with greater efficiency, as a follow-up to the ordinary
$f-k$ spectrum. Since a high-resolution array process
by design is intended to separate signals otherwise unre-
solvable, there is sound justification to limit its use to
the vicinity of signals tentatively identified beforehand
by less powerful but faster techniques. This is an advan-
tageous circumstance, since an M-signal test is a function
of $2M$ dimensions of wavenumber and would otherwise prove
computationally less efficient. Applying the 2-signal
test to the highest peak of an ordinary $f-k$ spectrum,
then, one hypothesizes the presence of 2 plane waves which
appear as 1 only because of their proximity. By the hypothe-
sis the spectral peak lies within the area of the main lobe
of either signal and thus $\in$ may be minimized directly by

the method of steepest descent. This is the procedure used
here.

Since, as has been shown earlier,

$$\vec{\bar{k}} = \vec{k}$$

is prohibited, the descent cannot begin from any one single

point in the $\mathbf{k}$-plane, as, for example, the peak under con-

sideration. But any pair of points in that vicinity is

suitable; all lead to the same solution. A convenient pair

are (1) the peak, and (2) the adjacent minimum of $\leqslant$ with

respect to, say, $\vec{\bar{k}}$ when $\vec{k}$ is fixed at the peak as in the

previously discussed figure . The gradient of $\leqslant$ is com-

puted at this pair and $\leqslant$ itself then recomputed at a new

location down the gradient. The length of this first step

in the descent is some fraction of the width of the array-

response main-lobe, thus chosen to ensure that the process

does not jump from the vicinity of the solution into the

range of an adjacent relative minimum. The gradient is new-

ly computed at this second location; another somewhat smaller

step is taken down the gradient; the gradient is once more

computed, now at this third location, and so forth in suc-

cessively smaller steps until the point is reached in that

4-dimensional space at which the gradient goes to zero.

Some examples with synthetic data of the LASA LP array follow.

The north-south half-width of the array-response main-lobe is about 0.0056 cycle/km. (figure   ) In the first example the input, U_n, consists of a signal at 0.0002 cycle/km north, and one at 0.0002 south. Thus the half-width of the main lobe is more than an order of magnitude greater than the distance between the signals in the k-plane. The signals are aqual, of unit size, their phase is equal (at the center of the array), and no noise is present. The ordinary $f-k$ spectrum, showing the merged signals with resultant solitary main lobe, and looking precisely like the array response, is given in figure   . With one vector fixed at the peak of this main-lobe while the other ranges the k -plane, $\in$ appears in the contoured plot of figure   . From this pair of points, i.e., the peak in figure   and the one in figure   , the descent is begun. Its progress and the final result are shown in the computer bulletin of figure   . This best fit precisely recovers the 2 signals: size, phase, and the wavenumber location to within less than 0.000005 cycles/km. The wavenumber distance between these 2 signals is only a degree or two of azimuth for Rayleigh waves.

In the second example, presented in similar format

(figures   and   ), the synthetic input consists of a
pair of signals in incoherent noise.

One signal, of unit size and zero phase, is located at
0.002 cycle/km south. Random numbers added to it reduce
the signal-to-noise ratio to 1. Finally, a very large signal,
100 times the size of the first, is located at 0.002 cycle/km
north. It is opposite in phase to the first one and thus it
interferes destructively with the small signal.

The location of the 2 signals, superposed on the array
response, is shown in figure   . The distance between them
is 0.7 the main-lobe half-width. The arrow indicates the
displacement of the smaller signal as recovered by the 2-signal
test. The small signal alone, in the presence of this same
noise sample, emerges with the same displacement (in the ordin-
ary $f-k$ spectrum). The steps in the descent to the solu-
tion, are presented in figure   . The distortion of amplitude
and phase of the large signal as recovered is about 1 percent;
that of the small signal, less than 5 percent. The incoher-
ent noise, of course, is the source of such distortion as is
present. The modeling process, being linear, separates plane
waves with fidelity, as demonstrated in the first example.

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